MATHEMATICS

B.Sc., Part III (paper V)

Uniform convergence (2)

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Cauchy's general Principle of uniform convergence.

Let (f_n) be a sequence of real valued functions defined on a set X. Then $\{f_n\}$ converges uniformly on X iff for every $\epsilon > 0$, $\exists$ a +ve integer m such that

$$n \geq m, p \geq m, x \in X \rightarrow |f_n(x) - f_p(x)| < \epsilon \quad \text{——} (1)$$

Proof. Necessity: suppose that the sequence (f_n) converges uniformly to the function f on X.

Then, given $\epsilon > 0$ $\exists$ a +ve integer m such that

$|f_n(x) - f(x)| < \epsilon/2 \ \forall n \geq m, x \in X$.

Hence, if $n, p \geq m$ we have, for any $x \in X$,

$$|f_n(x) - f_p(x)| = |f_n(x) - f(x) + f(x) - f_p(x)|$$
$$\leq |f_n(x) - f(x)| + |f(x) - f_p(x)|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$ Hence (1) holds for this m.

Sufficiency Let (f_n) be any sequence of real valued functions defined on X such that given $\epsilon > 0$, $\exists$ a +ve integer m such that (1) holds.

We have to show that $\exists$ a function f on X such that (f_n) converges uniformly to f on X.

From (1), we see that for each fixed $x \in X$ the sequences of real numbers $\{f_n(x)\}$ is a Cauchy sequence.

Hence $\lim_{n\to\infty} f_n(x)$ exists for every $x \in X$.

We define f by

$$f(x) = \lim_{n\to\infty} f_n(x), \forall x \in X.$$

Keeping p fixed in (1) and letting $n \to \infty$, we get

$|f(x) - f_p(x)| < \epsilon$ for all $p \geq m$ and all $x \in X$.

It follows that the sequence $\{f_n(x)\}$ converges uniformly to f on X.

Theorem (M_n-test): Let $\{f_n\}$ be a sequence of real valued functions defined on a set X.
Let $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$
and let $M_n = \sup\{|f_n(x) - f(x)| : x \in X\}$. Then $\{f_n\}$ converges uniformly to f iff $M_n \to 0$ as $n \to \infty$.

Proof (Necessity). Let $\{f_n\}$ converge uniformly to f on X.
Then for every $\epsilon > 0$ $\exists$ a +ve integer m, independent of x such that

$$n \geq m,\ x \in X \to |f_n(x) - f(x)| < \epsilon \quad \text{———} (1)$$

Since M_n is the supremum of $|f_n(x) - f(x)|$ for varying x, it follows from (1) that

$$n \geq m \to M_n \leq \epsilon.$$

Hence $M_n \to 0$ as $n \to \infty$.

(Sufficiency). Let $M_n \to 0$ as $n \to \infty$. Then, given $\epsilon > 0$, $\exists$ a +ve integer m such that

$$n \geq m \to M_n < \epsilon$$

i.e. $n \geq m \to \sup_{x \in X} |f_n(x) - f(x)| < \epsilon$.

Hence $|f_n(x) - f(x)| < \epsilon$ for all $n \geq m$ & $x \in X$. It follows that $\{f_n\}$ converges uniformly to f on X.

EXAMPLE Show that the sequence $\{f_n\}$ where $f_n(x) = nx(1-x)^n$ does not converge uniformly on $[0,1]$.

Solution: When $0 < x < 1$, we have $\lim_{n\to\infty} f_n(x)$

$$= \lim_{n\to\infty} \frac{nx}{(1-x)^{-n}} \left[\text{Form } \frac{\infty}{\infty}\right] = \lim_{n\to\infty} \frac{x}{-(1-x)^{-n}\log(1-x)}$$

$$= \lim_{n\to\infty} \frac{-x(1-x)^n}{\log(1-x)} = 0 \text{ since } (1-x)^n \to 0 \text{ as } n \to \infty.$$

$\therefore f(x) = \lim_{n\to\infty} f_n(x) = 0$ when $0 < x < 1$.

Also $f_n(x) = 0$ when $x = 0$ or 1.

Hence $f(x) = 0\ \forall x \in [0,1]$. Thus f is continuous on $[0,1]$.

Now $M_n = \sup_{x \in [0,1]} |f_n(x) - f(x)|$

$$= \sup\{nx(1-x)^n : x \in [0,1]\}$$

$$\geq n \cdot \tfrac{1}{n}\left(1 - \tfrac{1}{n}\right)^n \left(\text{taking } x = \tfrac{1}{n} \in [0,1]\right)$$

$$= \left(1 - \tfrac{1}{n}\right)^n \to \frac{1}{e} \text{ as } n \to \infty$$

Hence by the above theorem $\{f_n\}$ does not converge uniformly on $[0,1]$. Here 0 is a point of non-uniform convergence since as $n \to \infty$ as $x \to 0$.

Theorem (Weierstrass's M-test). A series $\sum_{n=1}^{\infty} u_n(x)$ of functions will converge uniformly on X if $\exists$ a convergent series $\sum_{n=1}^{\infty} M_n$ of +ve constants such that $|u_n(x)| \leq M_n$ for all n and all $x \in X$.

Proof: Since $\sum_{n=1}^{\infty} M_n$ is cgt, for a given $\epsilon > 0$, $\exists$ a +ve integer m such that

$$n \geq m \rightarrow M_{n+1} + M_{n+2} + \cdots + M_{n+p} < \epsilon \quad \text{———} (1)$$

for $p = 1, 2, 3 \cdots$

Also $|u_n(x)| \leq M_n$ ——— (2)

From (1) and (2), we have

$$|u_{n+1}(x) + u_{n+2}(x) + \cdots + u_{n+p}(x)|$$
$$\leq |u_{n+1}(x)| + |u_{n+2}(x)| + \cdots + |u_{n+p}(x)|$$
$$\leq M_{n+1} + M_{n+2} + \cdots + M_{n+p}$$
$$< \epsilon$$

for every $n \geq m$ & every $x \in X$.

Hence $\sum u_n(x)$ converges uniformly (also absolutely) on X.

EXAMPLE

Show that the series $\sum \frac{\cos nx}{n^2} = \cos x + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots$ converges uniformly on the real line R.

Solution, We have $\left|\frac{\cos nx}{n^2}\right| \leq \frac{1}{n^2} \; \forall x \in R$.

But the series $\sum \frac{1}{n^2}$ is known to be cgt.

Hence by Weierstrass's M-test, the given series converges uniformly on R.

EXAMPLE Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2 x} \text{ is uniformly convergent in } [1, \infty[.$$

Solution Here $u_n(x) = \frac{1}{1+n^2 x}$

$\therefore |u_n(x)| \leq \frac{1}{1+n^2} \; \forall x \in [1\ \infty[\; < \frac{1}{n^2}$

But $\sum \frac{1}{n^2}$ is known to be cgt.

Hence by Weierstrass's M-test the given series is uniformly cgt on $[1\ \infty[$.

THEOREM

Abel's Test: The series $\sum u_n(x) v_n(x)$ will converge uniformly in $[a,b]$ if

(i) $\sum u_n(x)$ is uniformly convergent in $[a,b]$,

(ii) The sequence $\{v_n(x)\}$ is monotonic for every $x \in [a,b]$,

(iii) $\{v_n(x)\}$ is uniformly bounded in $[a,b]$ i.e. $\exists$ a +ve number h, independent of x and n such that $|v_n(x)| < h$ for every value of x in $[a,b]$ and every +ve integer n.

Proof Let $R_{n,p}(x) = u_{n+1}(x)v_{n+1}(x) + \cdots + u_{n+p}(x)v_{n+p}(x)$

and $r_{n,p}(x) = u_{n+1}(x) + \cdots + u_{n+p}(x)$

Then $R_{n,p}(x) = r_{n,1}(x)v_{n+1}(x) + [r_{n,2}(x) - r_{n,1}(x)]v_{n+2}(x) +$

$\cdots + [r_{n,p}(x) - r_{n,p-1}(x)]v_{n+p}(x)$

$= r_{n,1}(x)[v_{n+1}(x) - v_{n+2}(x)] \cdots + r_{n,p-1}(x)[v_{n+p-1}(x) - v_{n+p}(x)]$

$+ r_{n,p}(x)v_{n+p}(x)$ ——— (1)

Now since $\{v_n(x)\}$ is monotonic,

$[v_{n+1}(x) - v_{n+2}(x)], [v_{n+2}(x) - v_{n+3}(x)], \cdots$

$[v_{n+p-1}(x) - v_{n+p}(x)]$,

All have the same sign for fixed values of x in $[a,b]$.

Also since $\{v_n(x)\}$ is uniformly bounded on $[a,b]$, we have

$|v_n(x)| < h$ ——— (2)

for all values of x in $[a,b]$ and every +ve integer n.

Again, since the series $\sum u_n(x)$ is uniformly cgt in $[a,b]$ for a given $\varepsilon > 0$, $\exists$ a +ve integer m independent of x, such that for $n \geq m$:

$|r_{n,p}(x)| = |u_{n+1}(x) + \cdots + u_{n+p}(x)| < \frac{\varepsilon}{3h}$ ——— (3)

For every value of x in $[a,b]$ and every +ve integer value of p,

we then have from (1) and (3),

$|R_{n,p}(x)| < \frac{\varepsilon}{3h}|v_{n+1}(x) - v_{n+2}(x)| + \frac{\varepsilon}{3h}|v_{n+2}(x) - v_{n+3}(x)|$

$+ \cdots + \frac{\varepsilon}{3h}|v_{n+p-1}(x) - v_{n+p}(x)| + \frac{\varepsilon}{3h}|v_{n+p}(x)|$ ——— (4)

$= \frac{\varepsilon}{3h}|v_{n+1}(x) - v_{n+p}(x)| + \frac{\varepsilon}{3h}|v_{n+p}(x)|$

Since $v_{n+1}(x) - v_{n+2}(x), v_{n+2}(x) - v_{n+3}(x)$, etc are of the same sign. Now $|v_{n+1}(x) - v_{n+p}(x)| \leq |v_{n+1}(x)| + |v_{n+p}(x)|$

$< h + h = 2h$ from (2)

Then (4) may be written as: $|R_{n,p}(x)| < \frac{\varepsilon}{3h} \cdot 2h + \frac{\varepsilon}{3h} h = \varepsilon$

for $n \geq$ and for every value of x in $[a,b]$.

$p = 1, 2, 3 \cdots$

Hence the series $\sum u_n(x) v_n(x)$ converges uniformly on $[a,b]$.